

Time evolution techniques for detectors in relativistic quantum information

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Abstract. The techniques employed to solve the interaction of a detector and a quantum field typically require perturbation methods. We introduce mathematical techniques to solve the time evolution of an arbitrary number of detectors interacting with a quantum field. Our techniques apply to harmonic oscillator detectors and can be generalized to treat detectors modeled by quantum fields. Since the interaction Hamiltonian we introduce is quadratic in creation and annihilation operators, we are able to draw from continuous variable techniques commonly employed in quantum optics.

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1. Introduction

The field of quantum information aims at understanding how to store, process, transmit, and read information efficiently exploiting quantum resources [1]. In the standard quantum information scenarios observers may share entangled states, employ quantum channels, quantum operations, classical resources and perhaps more advanced devices such as quantum memories and quantum computers to achieve their goals. In order to implement any quantum information protocol, all parties must be able to *locally* manipulate the resources and systems which are being employed. Although quantum information has been enormously successful at introducing novel and efficient ways of processing information, it still remains an open question whether relativistic effects can be used to enhance current quantum technologies and give rise to new relativistic quantum protocols.

The novel and exciting field of relativistic quantum information has recently gained increasing attention within the scientific community. An important aim of this field is to understand how the state of motion of an observer and gravity affects quantum information tasks. For a review on developments in this direction see [2]. Recent work has focussed on developing mathematical techniques to describe localized quantum

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fields to be used in future relativistic quantum technologies. The systems under investigation include fields confined in moving cavities [3] and wave-packets [4] [5]. Moving cavities in spacetime can be used to generate observable amounts of bipartite and multipartite entanglement [6, 7]. Interestingly, it was shown that the relativistic motion of these systems can be used to implement quantum gates [8], thus bridging the gap between relativistic-induced effects and quantum information processing. In particular, references [7, 8] employed covariance matrix formalism within the framework of continuous variables and showed that most of the gates necessary for universal quantum computation could be obtained by simply moving the cavity through especially tailored trajectories [9].

A third local system that has been considered for relativistic quantum information processing is the well known Unruh-deWitt detector [10], a point-like quantum system which follows a classical trajectory in spacetime and interacts locally with a global free quantum field. Such system has been employed with different degrees of success in a variety of scenarios, such as in the work unveiling the celebrated Unruh effect [10] or to extract entanglement from the vacuum state of a bosonic field. Unruh DeWitt detectors seem convenient for relativistic quantum information processing however, the mathematical techniques involved, namely perturbation theory, become extremely difficult to handle even for simple quantum information tasks such as teleportation [11]. The main aim of our research program is to develop detector models which are mathematically simpler to treat so they can be used in relativistic quantum information tasks. A first step in this direction was taken in [12] where a model to treat analytically a finite number of harmonic oscillator detectors interacting with a finite number of modes was proposed exploiting techniques from continuous variables. The covariance matrix formalism was employed to study the Unruh effect and extraction of entanglement from quantum fields without perturbation theory. The techniques introduced in [12] are restricted to simple trajectories in which the time evolution is trivial. To show in detail how the formalism introduced was applied, the authors presented over-simplified examples using detectors coupled to a single mode of the field which is formally only applicable when the field can be decomposed into a discrete set of modes with large frequency separation. This situation occurs, for example, when the detectors are inside a cavity. The detector model introduced in this work generalizes the model presented in [12] to include situations in which the time evolution is non-trivial.

We introduce the mathematical techniques required to solve the time evolution of a detector which couples to an arbitrary time-dependent frequency distribution of modes. The detector is assumed to be again a harmonic oscillator. The interaction of the detector with the field is purely quadratic in the operators and, therefore, we can employ the formalism of continuous variables taking advantage of the powerful mathematical techniques that have been developed in the past decade [2]. Our techniques allow us to obtain the explicit time dependent expectation value of relevant observables, such as mean excitation number of particles. As a concrete example, we employ our model to analyze the response of a detector, which moves along a arbitrary trajectory and is

coupled to a time-dependent frequency distribution of field modes. Recently it was shown that a spatially dependent coupling strength can be engineered to couple a detector to a gaussian distribution of frequency modes [13]. Here we analyze the case where the coupling strength varies in space and time in such way that the detectors effectively couples to a time evolving frequency distribution of plane waves that can be described by a single mode. This technique simplifies the Hamiltonian and an exact expression for the number operators can be found. We also discuss the extent of the impact of the techniques developed in this paper: in particular, we stress that they can be successfully applied for a finite number detectors following arbitrary trajectories. The formalism is also applicable when the detectors are confined within cavities. In this last case, the complexity of our techniques further simplifies due to the discrete structure of the energy spectrum. Finally, we note that the model can be generalized to the case where the detector is a quantum field itself.

2. Interacting systems for relativistic quantum information processing

Work in the field of relativistic quantum information has typically considered the standard point-like Unruh-DeWitt interaction between a detector and a field [10, 14, 15]. In this scenario the detector interacts with equal strength with all the modes of the field. In general, the exact equations that govern the dynamics of the system are difficult to solve. Therefore, perturbation theory techniques are employed to evolve the system. Here we present a novel method to treat the exact dynamics of the interaction using methods from symplectic geometry.

The general interaction Hamiltonian $H_I(t)$ between a quantum mechanical system (detector) interacting with a bosonic quantum field $\Phi(t, \mathbf{x})$ in 4-dimensional spacetime is

$$H_I(\tau) = m(\tau) \int d^3x \sqrt{-g} \mathcal{F}(\tau, \mathbf{x}) \Phi(\tau, \mathbf{x}), \quad (1)$$

where (τ, \mathbf{x}) are a suitable choice of coordinates for this spacetime, $m(\tau)$ is the monopole moment of the detector and g denotes the determinant of the metric tensor [16]. The function $\mathcal{F}(\tau, \mathbf{x})$ is the effective interaction strength between the detector and the field. When written in momentum space, it describes how the internal degrees of freedom of the detector couple to a time dependant distribution of the field modes. A specific example will be given in section 5. Such details can be determined by a particular physical model of interest. A more detailed and complete treatment of the interaction Hamiltonian (1) can be found in [13, 17, 18].

We consider a bosonic quantum field Φ which can be expanded in terms of a particular set of solutions to the field equations $\phi_{\mathbf{k}}(\tau, \mathbf{x})$ as

$$\Phi = \sum_{\mathbf{k}} [D_{\mathbf{k}} \phi_{\mathbf{k}} + \text{h.c.}], \quad (2)$$

where the variable \mathbf{k} is a set of discrete parameters and $D_{\mathbf{k}}$ are bosonic operators that satisfy the canonical commutation relations $[D_{\mathbf{k}}, D_{\mathbf{k}'}^\dagger] = \delta_{\mathbf{k}\mathbf{k}'}$. We refer to the solutions

$\phi_{\mathbf{k}}$ as field modes. We emphasize that the modes $\phi_{\mathbf{k}}$ need *not* be standard solutions to the field equations (i.e. plane waves in the case of a scalar field in Minkowski spacetime) but can also be wave-packets formed by linear superpositions of plane waves. The modes are all normalized according to the appropriate inner product [19].

We can engineer the function $F(\tau, \mathbf{x})$ such that

$$\int dV F(\tau, \mathbf{x}) \Phi(\tau, \mathbf{x}) = h(\tau) D_{\mathbf{k}_*} + \text{h.c.}, \quad (3)$$

where one mode, labelled via \mathbf{k}_* , has been selected out of the set $\{\phi_{\mathbf{k}}\}$, which in turn implies

$$H_I(\tau) = m(\tau) [h(\tau) D_{\mathbf{k}_*} + \text{h.c.}]. \quad (4)$$

Therefore, the coupling strength has been specially designed to make the detector couple to a single mode. In the case of a free massless 1 + 1-dimensional scalar field, the mode the detector couples to corresponds to a time dependent frequency distributions of plane waves. More details about this decomposition and how to interpret the modes as frequency distributions are available in [4]. Note that for the case of a field contained within a cavity, where the modes are explicitly discrete, our methods apply in a simple manner.

In the following, we explain how to derive the time evolution of an arbitrary number of detectors interacting with an arbitrary number of fields when the interaction Hamiltonian is of a purely quadratic form given by equation 4.

3. Time evolution of N interacting bosonic systems

The generalization of the quadratic Hamiltonian given by equation (4) to N interacting bosons is

$$H(t) = \sum_{j=1}^{N(2N+1)} \lambda_j(t) G_j, \quad (5)$$

where the functions λ_j are real and the operators G_j are Hermitian and quadratic in creation and annihilation operators $\{(d_j, d_j^\dagger)\}$. For example, $G = d_1^\dagger d_2^\dagger + d_1 d_2$. The operators G_i form a closed Lie algebra with Lie bracket $[G_i, G_j] = \epsilon_{ijk} G_k$. The complex numbers ϵ_{ijk} are called structure constants and form a completely antisymmetric matrix.

We wish to find the time evolution of our interacting system. In the general case, the Hamiltonian $H(t)$ does not commute with itself at different times $[H(t), H(t')] \neq 0$. Therefore, the time evolution is induced by the unitary operator

$$U(t) = \overleftarrow{\mathcal{T}} e^{-i \int_0^t dt' H(t')} \quad (6)$$

where $\overleftarrow{\mathcal{T}}$ stands for time ordering. We can employ techniques from Lie algebra and symplectic geometry [20, 21] to explicitly find a solution to equation (6). The unitary evolution of the Hamiltonian can be written as [22],

$$U(t) = \prod_j U_j(t) = \prod_j e^{-i F_j(t) G_j} \quad (7)$$

where the functions $F_j(t)$ associated with generators G_j are real and depend on time. By equating (6) and (7) and differentiating with respect to time we obtain

$$H(t)U(t) = \dot{F}_1 G_1 \prod_j U_j(t) + e^{-iF_1(t)G_1} \dot{F}_2(t) G_2 \prod_{j \geq 2} U_j(t) + \dots \quad (8)$$

Multiplying on the right by $U^{-1}(t)$ gives us a sum of similarity transformations

$$H(t) = \dot{F}_1(t)G_1 + \dot{F}_2(t)U_1 G_2 U_1^{-1} + \dot{F}_3(t)U_1 U_2 G_3 U_2^{-1} U_1^{-1} + \dots \quad (9)$$

We have obtained a set of $N(2N+1)$ coupled, non-linear, first order ordinary differential equations of the form

$$\sum_j \alpha_{ij}(t) \dot{F}_j(t) + \sum_j \beta_{ij} F_j(t) + \gamma_i(t) = 0. \quad (10)$$

The $\alpha_{ik}(t)$ and $\beta_{ik}(t)$ can be functions of $F_j(t)$ and $\lambda_j(t)$. The Hamiltonian and the initial conditions $F_j(0) = 0$ completely determine the unitary time evolution operator (6).

The equations can be re-written in a formalism which simplifies calculations by defining a vector that collects bosonic operators

$$\mathbb{X} := \left(d_1, d_1^\dagger, \dots, d_N, d_N^\dagger \right)^T \quad (11)$$

In this formalism the successive applications of the Baker-Campbell-Hausdorff formula which are required in the similarity transformations will be replaced by simple matrix multiplications reducing the problem from a tedious Hilbert space computation to simple linear algebra. We write

$$U_j(t) G_k U_j^{-1}(t) = \mathbb{X}^\dagger \cdot \mathbf{S}_j(t)^\dagger \cdot \mathbf{G}_j \cdot \mathbf{S}_j(t) \cdot \mathbb{X} \quad (12)$$

where

$$U_j(t) \mathbb{X} U_j^{-1}(t) := \mathbf{S}_j(t) \cdot \mathbb{X}$$

and \mathbf{G}_j is the matrix representation of G_j defined via $G_j := \mathbb{X}^\dagger \cdot \mathbf{G}_j \cdot \mathbb{X}$. The dynamical transformation of the vector of operators \mathbb{X} generated by the interaction Hamiltonian G_j is given by the symplectic matrix [23]

$$\mathbf{S}_j := e^{-iF_j(t)\mathbf{\Omega}\mathbf{G}_j} \quad (13)$$

where $F_j(t)$ are real functions associated with the generator G_j and $\Omega_{ij} := [\mathbb{X}_i, \mathbb{X}_j]$ is the symplectic form. The symplectic matrix \mathbf{S} satisfies $\mathbf{S}^\dagger \mathbf{\Omega} \mathbf{S} = \mathbf{\Omega}$. In this formalism equation (9) takes the form

$$\mathbb{X}^\dagger \cdot \mathbf{H} \cdot \mathbb{X} = \dot{F}_1(t) \mathbb{X}^\dagger \cdot \mathbf{G}_1 \cdot \mathbb{X} + \dot{F}_2(t) \mathbb{X}^\dagger \cdot \mathbf{S}_1(t)^\dagger \cdot \mathbf{G}_2 \cdot \mathbf{S}_1(t) \cdot \mathbb{X} + \dots$$

and therefore, the matrix representation of the Hamiltonian \mathbf{H} is given by

$$\begin{aligned} \mathbf{H} = & \dot{F}_1(t) \mathbf{G}_1 + \dot{F}_2(t) \mathbf{S}_1(t)^\dagger \cdot \mathbf{G}_2 \cdot \mathbf{S}_1(t) \\ & + \dot{F}_3(t) \mathbf{S}_1(t)^\dagger \cdot \mathbf{S}_2(t)^\dagger \cdot \mathbf{G}_3 \cdot \mathbf{S}_2(t) \cdot \mathbf{S}_1(t) + \dots \end{aligned} \quad (14)$$

The matrix products of the form

$$\dot{F}_j(t) \mathbf{S}_k(t)^\dagger \cdot \mathbf{G}_j \cdot \mathbf{S}_k(t) \quad (15)$$

in equation (14) must be explicitly computed such that it is possible to re-write equation (14) in terms of the generators \mathbf{G}_i . By equating the coefficients of equation (14) to the coefficients $\lambda_j(\tau)$ in equation (5) we obtain a set of coupled $N(2N + 1)$ ordinary differential equations. Solving for the functions $F_j(t)$, we obtain the explicit expression for the time evolution of the system as described by equation (7). The final expression is

$$\mathbf{S}(t) = \prod_j \mathbf{S}_j(t) = \prod_j e^{-iF_j(t)\Omega\mathbf{G}_j}, \quad (16)$$

which corresponds to the time evolution of the whole system. A case of great interest is that of Gaussian states which are common in quantum optics and relativistic quantum theory [2]. In this case the state of the system is encoded by the first moments $\langle \mathbb{X}_j \rangle$ and a covariance matrix $\mathbf{\Gamma}(t)$ defined by

$$\Gamma_{ij} = \langle \mathbb{X}_i \mathbb{X}_j + \mathbb{X}_j \mathbb{X}_i \rangle - 2\langle \mathbb{X}_i \rangle \langle \mathbb{X}_j \rangle. \quad (17)$$

The time evolution of the state is given by

$$\mathbf{\Gamma}(t) = \mathbf{S}^\dagger(t)\mathbf{\Gamma}(0)\mathbf{S}(t). \quad (18)$$

4. Application: Time evolution of a detector coupled to a field

We now apply our formalism to describe a situation of great interest to the field of relativistic quantum information: a single detector following a general trajectory and interacting with quantum field via a general time and space dependent coupling strength. For simplicity, consider an uncharged massless scalar field $\Phi(t, x)$ in $1 + 1$ dimensional Minkowski spacetime with coordinates (t, x) and metric $g_{\mu\nu} = (-1, 1)$. The field obeys the standard Klein-Gordon equation $\square\Phi = 0$ where $\square := -\partial_t^2 + \partial_x^2$ is the d'Alembertian in flat Minkowski coordinates. The plane-wave solutions to the field equation are

$$\phi_\omega(t, x) = \frac{1}{2\pi\sqrt{\omega}} e^{-i\omega(t-x)} \quad (19)$$

which are (Dirac delta) normalized $(\phi_\omega, \phi_{\omega'}) = \delta(\omega - \omega')$ through the standard conserved inner product (\cdot, \cdot) , see [19]. The field can be expanded in terms of these solutions as

$$\Phi = \int_0^{+\infty} d\omega [a_\omega \phi_\omega(t, x) + a_\omega^\dagger \phi_\omega^*(t, x)], \quad (20)$$

The annihilation operators a_ω define the well known Minkowski vacuum $|0\rangle_M$ through

$$a_\omega |0\rangle_M = 0, \quad \forall \omega > 0. \quad (21)$$

The degrees of freedom of the detector which we assume to be a harmonic oscillator are described by the bosonic operators d, d^\dagger that satisfy the usual commutation relations $[d, d^\dagger] = 1$. The vacuum $|0\rangle_d$ of the detector is defined by $d|0\rangle_d = 0$. Therefore, the vacuum $|0\rangle$ of the *non-interacting* theory takes the form $|0\rangle := |0\rangle_d \otimes |0\rangle_M$.

In the interaction picture, we assume that the detector couples to the field via the interaction Hamiltonian

$$H_I(t) = m(t) \int dx \sqrt{-g} \mathcal{F}(t, x) \int_0^{+\infty} d\omega [a_\omega \phi_\omega(t, x) + a_\omega^\dagger \phi_\omega^*(t, x)], \quad (22)$$

where $\mathcal{F}(t, x)$ is a time and space dependent coupling strength and g denotes the determinant of the metric tensor. In momentum space we can see that the detector couples to a time dependent frequency distribution of plane waves

$$H_I(t) = m(t) \int d\omega \mathcal{G}(t, \omega) e^{i\omega t} \quad (23)$$

where

$$\mathcal{G}(\tau, \omega) = \int dx \sqrt{-g} F(\tau, x) e^{-i\omega x} \quad (24)$$

is the frequency window of the detector which in general evolves in time. We now parameterise the interaction via a suitable set of coordinates, (τ, ξ) , that describe a frame comoving with the detector [17, 18].

$$H_I(\tau) = m(\tau) \int d\xi \sqrt{-g} \mathcal{F}(\tau, \xi) \int_0^{+\infty} d\omega [a_\omega \phi_\omega(\tau, \xi) + a_\omega^\dagger \phi_\omega^*(\tau, \xi)], \quad (25)$$

In this comoving frame, the monopole moment of the detector is

$$m(\tau) = s(\tau) [e^{-i\Delta\tau} d + e^{i\Delta\tau} d^\dagger] \quad (26)$$

$s(\tau)$ is real function which allows us to turn on and off the interaction and Δ is frequency of the harmonic oscillator. In momentum space the detector couples to a time-dependent frequency distribution of Minkowski plane-wave field modes. In [13] the spatial dependence of the coupling strength was specially designed to couple the detector to peaked distributions of Minkowsky or Rindler modes. Here we consider a coupling strength that can be designed to couple the detector to a time-varying wave-packet. It is therefore more convenient to decompose the field not in the plane-wave basis but in a special decomposition

$$\Phi = D\psi + D^\dagger\psi^* + \Phi', \quad (27)$$

where ψ is the mode the detector couples to, which corresponds to a time dependent frequency distribution of plane waves. The operators D, D^\dagger are time independent and satisfy the canonical commutation relations $[D, D^\dagger] = 1$. The field Φ' includes all the modes orthogonal to ψ and we will assume them to be countable. Therefore

$$h(\tau) = \int d\xi \mathcal{F}(\tau, \xi) \psi(\tau, \xi), \quad \int d\xi \mathcal{F}(\tau, \xi) \Phi'(\tau, \xi) = 0 \quad \forall \tau. \quad (28)$$

Such a decomposition can always be formed from a complete orthonormal basis (an example of which can be found in [17]). In general, the operator D does not annihilate the Minkowski vacuum $|0\rangle_M$. This observation is a consequence of fundamental ideas that lie at the foundation of quantum field theory, where different and inequivalent definitions of particles can coexist. Such concepts are, for example, at the very core of the Unruh effect [10]. The operator D will annihilate the vacuum $|0\rangle_D$. Note that the vacuum state $|0\rangle_I$ of this interacting system is different from the vacuum state $|0\rangle$ of the noninteracting theory, i.e. $|0\rangle \neq |0\rangle_I$.

In this decomposition, the interaction Hamiltonian takes a very simple form

$$H_I(\tau) = m(\tau) \cdot [h(\tau)D + h^*(\tau)D^\dagger] \quad (29)$$

which describes the effective interaction between the internal degrees of freedom of a detector following a general trajectory and coupling to a *single mode* of the field described by D . The time evolution of the system can be solved in this case by employing the techniques we introduced in the previous section. However, this formalism is directly applicable to describe the interaction of N detectors with the field. In that case, our techniques yield differential equations which can be solved numerically. We choose here to demonstrate our techniques with the single detector case since it is possible to compute a simple expression to the expectation value of the number of particles in the detector.

Let the detector-field system be in the ground state $|0\rangle_D$ at $\tau = 0$. We design a coupling such that we obtain an interaction of the form (29). In this case, the covariance matrix only changes for the detector and our preferred mode. The subsystem described by d, D is always separable from the rest of the non-interacting modes. The covariance matrix of the vacuum state $|0\rangle_D$ is represented by the 4×4 identity matrix, i.e. $\mathbf{\Gamma}(0) = \mathbf{1}$. The final state $\mathbf{\Gamma}(\tau)$ therefore takes the simple form of $\mathbf{\Gamma}(\tau) = \mathbf{S}^\dagger \mathbf{S}$. The final state provides the information we need to compute the time dependent expectation value $N_d(\tau) := \langle d^\dagger d \rangle(\tau)$.

From the definition of the covariance matrix $\mathbf{\Gamma}(\tau)$, one finds that $N_d(\tau)$ is related to $\mathbf{\Gamma}(\tau)$ by

$$\mathbf{\Gamma}_{11}(\tau) = 2 \langle d^\dagger d \rangle(\tau) - 2 \langle d^\dagger \rangle(t) \langle d \rangle(\tau) + 1 \quad (30)$$

In this paper we choose to work with states that have first moments zero, i.e. $\langle X_j \rangle = 0$. In this case, since our interaction is quadratic, the first moments will remain zero [24]. Therefore we are left with equation

$$\mathbf{\Gamma}_{11}(\tau) = 2 \langle d^\dagger d \rangle(\tau) + 1, \quad (31)$$

Our expressions hold for detectors moving along an arbitrary trajectory and coupled to an arbitrary wave-packet. Given a scenario of interest, one can solve the differential equations, obtain the functions $F_j(\tau)$ and, by using the decomposition in equation (16), one can obtain the time evolution of the system. We can then find the expression for the average number of excitations in the detector at time τ , which reads

$$N_d(\tau) = \frac{\cosh(2F_1(\tau)) \cosh(2F_2(\tau)) \cosh(2F_3(\tau)) \cosh(2F_4(\tau)) - 1}{2}. \quad (32)$$

For this choice of initial state we find that the functions $F_j(\tau)$ are associated with the generators $G_1 = d^\dagger D^\dagger + dD$, $G_2 = -i(d^\dagger D^\dagger - dD)$, $G_3 = d^{\dagger 2} + d^2$, $G_4 = -i(d^{\dagger 2} - d^2)$, respectively. The appearance of these functions can be simply related to the physical significance of the operators G_j . In fact, the generators G_1 and G_2 are related to the well known two-mode squeezing operators. Such operations generate entanglement and are known to break particle number conservation. The two generators G_3 and G_4 are related to the single-mode squeezing operators for the mode d . The generators $G_1 \dots G_4$, together with the generators $G_5 = D^{\dagger 2} + D^2$ and $G_6 = -i(D^{\dagger 2} - D^2)$ which represent single mode squeezing for the mode D , form the set of so-called active transformations of a gaussian state and do not conserve total particle number. The

remaining operators, whose functions are absent in equation (32), form the passive transformations for gaussian states. These operators are also known as generalised beam splitter [23]; they conserve the total particle number of a state and hence do not contribute to equation (32).

It is also of great physical interest to study the response of one (or more) detector when the initial state is a different vacua $|\tilde{0}\rangle$, for example the Minkowski vacuum $|0\rangle_M$. It is well known [16] that the relation between different vacua, for example $|\tilde{0}\rangle$ and $|0\rangle_D$, is

$$|\tilde{0}\rangle = N e^{-\frac{1}{2} \sum_{ij} V_{ij} D_i^\dagger D_j^\dagger} |0\rangle_D, \quad (33)$$

where N is a normalization constant and the symmetric matrix \mathbf{V} is related to the Bogoliubov transformations between two different set of modes $\{\phi_\omega\}, \{\psi_j\}$ used in the expansion of the field. The modes $\{\phi_\omega\}$ are related to the vacuum $|\tilde{0}\rangle$ while the modes $\{\psi_j\}$ are related to the vacuum $|0\rangle_D$. In general, the matrix \mathbf{V} takes the form $\mathbf{V} := \mathbf{B}^* \mathbf{A}^{-1}$ [16], where the matrices \mathbf{A} and \mathbf{B} collect the Bogoliubov coefficients $A_{j\omega}, B_{j\omega}$ which for uncharged scalar fields are defined as

$$A_{j\omega} = (\psi_j, \phi_\omega) \quad (34)$$

$$B_{j\omega} = -(\psi_j, \phi_\omega^*) \quad (35)$$

In the covariance matrix formalism, we collect the detector operators d, d^\dagger and mode operators D_i, D_i^\dagger in the vector $\mathbb{X} := (d, d^\dagger, D_1, D_1^\dagger, D_2, D_2^\dagger, \dots)^T$. The initial state $\mathbf{\Gamma}(0)$ will not be the identity anymore, $\mathbf{\Gamma}(0) = \mathbf{\Delta} \neq \mathbf{1}$. The final state $\mathbf{\Gamma}(\tau)$ will take the form

$$\mathbf{\Gamma}(\tau) = \mathbf{S}^\dagger(\tau) \mathbf{\Delta} \mathbf{S}(\tau) \quad (36)$$

We assume that within the field expansion using the basis $\{\psi_j\}$, the detector interacts only with the mode ψ_p . We can again apply our techniques to obtain the number expectation value as

$$\begin{aligned} N_d(t) = & S_{12}^\dagger S_{12} + S_{1(2+p)}^\dagger S_{1(2+p)} (\mathbf{V} \mathbf{V}^\dagger)_{pp} \\ & + S_{1(3+p)}^\dagger S_{1(3+p)} \left(1 + (\mathbf{V} \mathbf{V}^\dagger)_{pp} \right) \\ & - S_{1(2+p)}^\dagger S_{1(3+p)} V_{pp}^* - S_{1(3+p)}^\dagger S_{1(2+p)} V_{pp}. \end{aligned} \quad (37)$$

5. Concrete Example: inertial detector interacting with a time-dependent wavepacket

To further specify our example we consider the detector stationary and interacting with a localized time dependent frequency distribution of plane waves. The free scalar field is decomposed into wave packets of the form [17]

$$\tilde{u}_{ml} := \int dk f_{ml}(k) u_k \quad (38)$$

where the distributions $f_{ml}(k)$ are defined as

$$f_{ml}(k) := \begin{cases} \epsilon^{-1/2} e^{-2i\pi l k / \epsilon} & \epsilon(m - 1/2) < k < \epsilon(m + 1/2) \\ 0 & \text{otherwise} \end{cases} \quad (39)$$

with $\epsilon > 0$ and $\{m, l\}$ running over all integers. The mode operators associated with these modes are defined as

$$a_{ml} := \int dk f_{ml}^*(k) a_k \quad (40)$$

The distributions $f_{ml}(k)$ satisfy the completeness and orthogonality relations

$$\sum_{ml} f_{ml}(k) f_{ml}^*(k') = \delta(k - k') \quad \int dk f_{ml}(k) f_{m'l'}^*(k) = \delta_{mm'} \delta_{ll'} \quad (41)$$

The wave packets are normalised as $(\tilde{u}_{ml}, \tilde{u}_{m'l'}) = \delta_{mm'} \delta_{ll'}$ and operators satisfy the commutation relations $[a_{ml}, a_{m'l'}^\dagger] = \delta_{mm'} \delta_{ll'}$. The scalar field can then be expanded in terms of these wave packets as

$$\Phi = \sum_{ml} [\tilde{u}_{ml} a_{ml} + h.c.] \quad (42)$$

Following [18], we consider an inertial detector and so we can parametrise our interaction via $t = \tau$ and $x = \xi$. We now construct our detectors spatial profile to be

$$\mathcal{G}(\tau, \xi) := h(\tau) \int dk f_{ML}^*(k) u_k(\tau, \xi) \quad (43)$$

where $h(\tau)$ is now an arbitrary time dependent function which dictates when to switch on and off the detector. Physically, this corresponds to a detector interaction strength that is changing in time to match our preferred mode, labelled by M, L . We point out that any other wave packet decomposition could be chosen as long as it satisfies completeness and orthogonality relations of the form (41). The form of the spatial profile to pick out these modes is therefore general. Inserting the profile (43) into our interaction Hamiltonian (25), we obtain

$$H_I(\tau) = (d e^{-i\Delta\tau} + d^\dagger e^{+i\Delta\tau}) \left(h(\tau) a_{ML} + h^*(\tau) a_{ML}^\dagger \right) \quad (44)$$

We choose the switching on function to be $h(\tau) = \tau^2 e^{-\tau^2/T}$, where T modulates the interaction time. The interaction Hamiltonian is then

$$H_I(\tau) = \tau^2 e^{-\tau^2/T} (d e^{-i\Delta\tau} + d^\dagger e^{+i\Delta\tau}) (a_{LM} + a_{LM}^\dagger) \quad (45)$$

which written in symplectic form (see equation (5)) is

$$H_I(\tau) = \tau^2 e^{-\tau^2/T} [\cos(\tau\Delta) G_1 + \sin(\tau\Delta) G_2 + \cos(\tau\Delta) G_7 + \sin(\tau\Delta) G_8] \quad (46)$$

where $G_7 = d^\dagger D + d D^\dagger$ and $G_8 = -i(d D^\dagger - d^\dagger D)$. The matrix representation of H_I is

$$\mathbf{H}_I = \frac{\tau^2 e^{-\tau^2/T}}{2} \begin{bmatrix} 0 & 0 & e^{-i\tau\Delta} & e^{i\tau\Delta} \\ 0 & 0 & e^{-i\tau\Delta} & e^{i\tau\Delta} \\ e^{-i\tau\Delta} & e^{i\tau\Delta} & 0 & 0 \\ e^{-i\tau\Delta} & e^{i\tau\Delta} & 0 & 0 \end{bmatrix} \quad (47)$$

Equating (47) and (14), or equivalently (46) and (5), gives us the ordinary differential equations we need to find the functions F_j for this specific example. Here we solve the

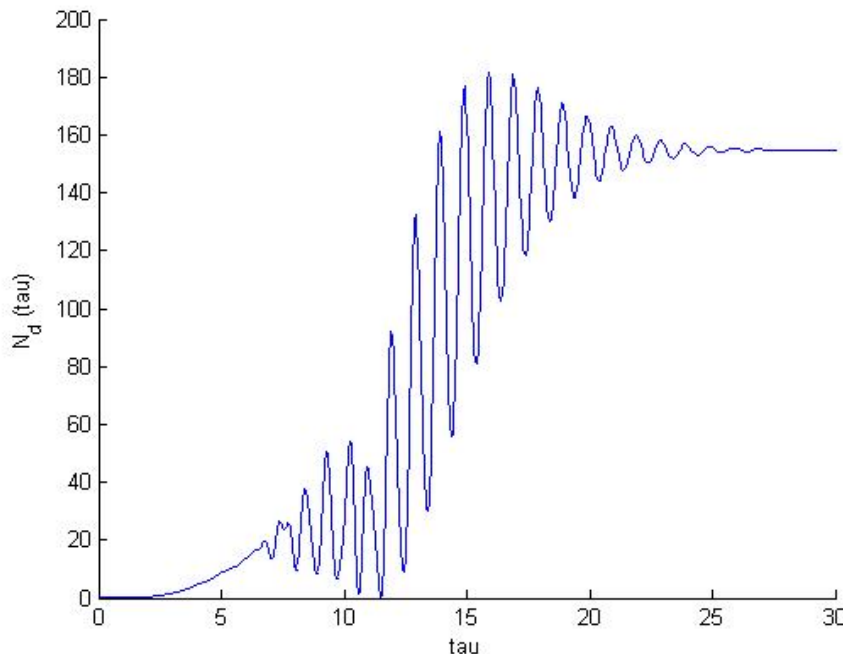


Figure 1. Mean number of particles, $n_d(\tau)$, as a function of time τ . Here we used $T = 80$ and $\Delta = 2\pi$.

equations for $F_j(t)$ numerically and we plot the average number of detector excitations $n_d(\tau)$ as a function of time in figure (5).

We find that the number expectation value of the detector grows and oscillates as a function of time while the detector and field are coupled. This can be expected since the time dependence of the Hamiltonian is multiplied by complex exponentials that will induce phase rotations in the state and hence oscillations in the number operator. Finally, the number expectation value reaches a constant value after the interaction is turned off. This is expected as well since, once the detector evolves freely, the free Hamiltonian does not account for emissions of particles from the detector.

6. Discussion

It is of great interest to solve the time evolution of interacting bosonic quantum systems since they are relevant to quantum optics, quantum field theory and relativistic quantum information, among many research fields. In most cases, it is necessary to employ perturbative techniques which assume a weak coupling between the bosonic systems. In relativistic quantum information, perturbative calculations used to study tasks such as teleportation [11] and extraction of vacuum entanglement [11] become very complicated already for two or three detectors interacting with a quantum field. In cases where the computations become involved, physically motivated or ad hoc approximations can aid, however, in most cases, powerful numerical methods must be invoked and employed to

study the time evolution of quantities of interest.

We have provided mathematical methods to derive the differential equations that govern the time evolution of N interacting bosonic modes coupled by a purely quadratic interaction. The techniques we introduce allow for the study of such systems beyond perturbative regimes. The number of coupled differential equations to solve is $N(2N+1)$, therefore making the problem only polynomially hard. Symmetries, separable subsets of interacting systems, among other situations can further reduce the number of differential equations.

The Hamiltonians our method is applicable to include a large class of interactions. In this paper, as a simple example, we have applied our mathematical tools to analyze the time evolution of a single harmonic oscillator detector interacting with a quantum field. However, our techniques are readily applied to N detectors following any trajectory while interacting with a finite number of wave-packets through an arbitrary interaction strength $\mathcal{F}(t, x)$. Our techniques simplify greatly when the detectors are confined within a cavity where the field spectrum becomes discrete. The cavity scenario allows one to couple a detector to a single mode of the field in a time independent way as, in principle, no discrete mode decomposition needs to be enforced. Therefore, the single mode interaction Hamiltonian (4) can arise in a straight forward fashion. Inside a cavity, the oversimplified examples introduced in [12] where two harmonic oscillators couple to a single mode of the field are well known to hold trivially.

We have further specified our example to analyze the case of an inertial detector interacting with a time-dependent wave-packet. We have showed how to engineer a coupling strength such that the interaction Hamiltonian can be described by an effective single field mode. However, the field mode is not a plane-wave but a time dependent frequency distribution of plane waves. In this case we have solved the differential equations numerically and showed the number of detector excitations oscillates in time while the detector is on.

Work in progress includes using these detectors to extract field entanglement and perform quantum information tasks.

Note

Near the completion of this work but before posting our results, we became aware of another group working independently along similar lines [25]. We agreed to post our results simultaneously.

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